MATH 430, SPRING 2022
NOTES APRIL 11-15

Theorem 1. *Suppose that $f : \mathbb{N}^k \to \mathbb{N}$ is primitive recursive. Then there is a Δ_1 formula $\phi(x_0, ..., x_{k-1}, y)$, such that for all $a_1, ..., a_{k-1}, b$ in $\mathbb{N}$,*

$$f(a_1, ..., a_{k-1}) = b \text{ iff } \mathfrak{A} \models \phi[a_1, ..., a_{k-1}, b].$$

Proof. (Sketch) Fix a primitive recursive function f, and suppose for simplicity that $f : \mathbb{N} \to \mathbb{N}$ (the general case is similar). Say $f(0) = d$ and for all n, $f(n+1) = g(f(n), n)$ here g is primitive recursive. By induction, we can assume that $g(x, y) = z$ is equivalent to a Δ_1 formula.

Below we are use the Chinese Remainder Theorem to talk about the sequence $\vec{c}$.

First we show that $f(x) = y$ can be represented by a Σ_1 formula: $f(n) = b$ iff there is a sequence $\vec{c} = \langle c_0, ...c_n \rangle$, such that $c_0 = d$ and for all $i < n$, $c_{i+1} = g(c_i, i)$ and $c_n = b$.

Next, we show that $f(x) = y$ can be represented by a Σ_1 formula: $f(n) = b$ iff for every sequence $\vec{c} = \langle c_0, ...c_n \rangle$, such that $c_0 = d$ and for all $i < n$, $c_{i+1} = g(c_i, i)$, we have that $c_n = b$.

It follows that $f(x) = y$ is equivalent to a Δ_1 formula. □

Corollary 2. *There is a Δ_1 formulas $\phi_{exp}(x, y, x)$, such that for all natural numbers a, b, c, $a^b = c$ iff $\mathfrak{A} \models \phi_{exp}[a, b, c]$.*

Proof. This is because $f(a, b) = a^b$ is primitive recursive and the above theorem. □

Definition 3. *The collection of* **partial recursive functions** *are all partial functions $f : \mathbb{N}^k \rightharpoonup \mathbb{N}$ build up from the primitive recursive functions, using composition and the "minimization" operation:*

if $g : \mathbb{N}^{k+1} \to \mathbb{N}$ is total recursive, and

$$f(x_1, ..., x_k) = \text{ least } y \text{ such that } g(x_1, ..., x_k, y) = 0,$$

then f is partial recursive.

Theorem 4. *If $f : \mathbb{N}^k \rightharpoonup \mathbb{N}$ is partial recursive, then there is a Σ_1 formula $\phi(x_0, ..., x_k)$, such that, for all $a_1, ..., a_{k-1}, b$ in $\mathbb{N}$,*

$$f(a_1, ..., a_{k-1}) = b \text{ iff } \mathfrak{A} \models \phi[a_1, ..., a_{k-1}, b].$$

Proof. By induction on the complexity of f. If f is primitive recursive, this follows from the above theorem.

Suppose that $f = g \circ h$, and by the inductive hypothesis, both g, h are equivalent to Σ_1 formulas ϕ_g, ϕ_h, respectively. Then $f(a_1, ..., a_k) = b$ iff

$g(h(a_1,...,a_k)) = b$ iff $\exists x(h(a_1,...,a_k) = x \wedge g(x) = b)$ iff $\exists x(\phi_h(a_1,...,a_k,x) \wedge \phi_g(x,b))$.

Note that the latter is Σ_1, since both ϕ_g, ϕ_h are Σ_1 and we only used an extra existential quantifier. □

Corollary 5. *If $f : \mathbb{N}^k \to \mathbb{N}$ is total recursive, then there is a Δ_1 formula $\phi(x_0,...,x_k)$, such that, for all $a_1,...,a_{k-1}, b$ in $\mathbb{N}$,*

$$f(a_1,...,a_{k-1}) = b \textit{ iff } \mathfrak{A} \models \phi[a_1,...,a_{k-1},b].$$

Proof. By the above theorem, there is a Σ_1 formula $\phi(x_0,...,x_k)$, such that, for all $a_1,...,a_{k-1}, b$ in $\mathbb{N}$, $f(a_1,...,a_{k-1}) = b$ iff $\mathfrak{A} \models \phi[a_1,...,a_{k-1},b]$

But then $f(a_1,...,a_{k-1}) \neq b$ iff $\exists c(c \neq b \wedge f(a_1,...,a_{k-1}) = c)$ iff there is $c \neq b$ such that $\mathfrak{A} \models \phi[a_1,...,a_{k-1},c]$ iff $\mathfrak{A} \models \exists c(c \neq b \wedge \phi[a_1,...,a_{k-1},b])$.

It follows that both ϕ and $\neg\phi$ are Σ_1, and this means that ϕ is Δ_1. □

Let $A \subset \mathbb{N}^k$; the **characteristic function of** A, χ_A is given by $\chi_A(x) = 0$ if $x \in A$ and $\chi_A(x) = 1$ if $x \notin A$. Next we give two equivalent definitions of a recursive set.

Definition 6. *A set $A \subset \mathbb{N}^k$ is* **recursive** *iff its characteristic function χ_A is recursive iff $A = \{\langle a_1,...,a_k\rangle \mid \mathfrak{A} \models \phi[a_1,...,a_k]\}$ for some Δ_1 formula ϕ.*

Examples of recursive sets: any finite set; the set of all prime numbers; the set of all triples $\langle a,b,c\rangle$ such that $a^b = c$.

Definition 7. *A set $A \subset \mathbb{N}^k$ is* **recursively enumerable (r.e.)** *iff $A = \{\langle a_1,...,a_k\rangle \mid \mathfrak{A} \models \phi[a_1,...,a_k]\}$ for some Σ_1 formula ϕ.*

The following two propositions are left as exercises.

Proposition 8. *Suppose $A \subset \mathbb{N}^k$. If both A and its complement $\mathbb{N}^k \setminus A$ are r.e, then A is recursive.*

Proposition 9. *Let $A \subset \mathbb{N}^k$.*

(1) *If A is the domain of a partial recursive function, then A is r.e.;*
(2) *If A is the range of a partial recursive function, then A is r.e.*

CODING SEQUENCES

Next we define how to code sequences in a primitive recursive way. First, we order the primes: $2, 3, 5, 7, 11, 13, 17, 19, 23, ...$ and label them $p_0, p_1, p_2, ...$. For example the least prime p_0 is 2; $p_1 = 3, p_2 = 5, p_8 = 23$, etc.

Proposition 10. *The function $f(n) = p_n$ is primitive recursive.*

Proof. We have to write f in a primitive recursive way. Recall the proof that there are infinitely many primes: Otherwise, if $p_0, ..., p_n$ enumerate all of the primes, we take $p_0 \cdot p_1 \cdot ... p_n + 1$ and argue it is a prime to get a contradiction. That means that if we know that the first n primes (counting from zero) are $p_0, ..., p_n$, if p is the next prime, then $p \leq p_0 \cdot p_1 \cdot ... p_n + 1 \leq p_n^{n+1} + 1$. So we have:

- $f(0) = 2$,
- $f(n+1) =$ the least prime $p \leq f(n)^{n+1} + 1$ such that $p > f(n)$.

This is a primitive recursive definition, since multiplication is primitive recursive and we are only doing a bounded search for the next prime. (If the search was unbounded, then f would be recursive but not necessarily primitive recursive.)

To be even more precise, let $g(a, n) =$ the least prime p such that $a < p \leq a^{n+1} + 1$. Since exponentiation, addition and computing if something is a prime are all primitive recursive, and we only use bounded quantifiers, g is also primitive recursive. Then

- $f(0) = 2$,
- $f(n+1) = g(f(n), n)$.

□

Definition 11. *Given a sequence of natural numbers* $\vec{a} = \langle a_0, a_1, ..., a_k \rangle$*, we* **code** $\vec{a}$ *by the number* $p_0^{a_0+1} \cdot p_1^{a_1+1} \cdot ... p_k^{a_k+1}$.

Note that not every number codes a sequence, but any two different sequence are coded by different numbers. Examples:

- The sequence $\langle 2, 1, 0 \rangle$ is coded by $2^{2+1} \cdot 3^{1+1} \cdot 5^{0+1} = 2^3 \cdot 3^2 \cdot 5 = 8 \cdot 9 \cdot 5 = 360$;
- The sequence $\langle 0 \rangle$ is coded by $2^{0+1} = 2$;
- The sequence $\langle 0, 0, 2 \rangle$ is coded by $2 \cdot 3 \cdot 5^3 = 6 \cdot 125 = 750$;
- The sequence $\langle 9, 0 \rangle$ is coded by $2^{9+1} \cdot 3^{0+1} = 1024 \cdot 3 = 3072$;

And here are some examples of numbers that Do Not code sequences: $7, 14, 100, 42$. Why? In order for a number a to code a sequence, if a prime p divides a, then all primes less than p must also divide a. For example, $14 = 2 \cdot 7$, the prime 7 divides it, but 3 and 5 do not. That is why 14 does not code a sequence.

Next we write down some formulas that will be useful later.

(1) Δ_0: $\phi_{div}(y, x)$ is such that: a divides b iff $\mathfrak{A} \models \phi_{div}[a, b]$;
(2) Δ_0: $\phi_{prime}(x)$ is such that: p is a prime iff $\mathfrak{A} \models \phi_{prime}[p]$;
 $\phi_{prime}(x)$ is $x > 1 \wedge \forall y < x(\phi_{div}(y, x) \to y = 1)$.
(3) Δ_1: $\phi_{exp}(a, b, c)$, where $a^b = c$ iff $\mathfrak{A} \models \phi_{exp}[a, b, c]$.
 This is because $f(a, b) = a^b$ is primitive recursive, and so it is equivalent to a Δ_1 formula.
(4) Δ_1: $\phi_{th-prime}(p, n)$ is such that: $p = p_n$ i.e. the n-th prime iff $\mathfrak{A} \models \phi_{th-prime}[n, p]$.
 We can find such a formula because $f(n) = p_n$ is primitive recursive.

(5) Δ_1: $\phi_{code}(x)$ is such that: a codes a sequence iff $\mathfrak{A} \models \phi_{code}[a]$.
$\phi_{code}(x)$ is
$\forall y \leq x \forall z < y([\phi_{prime}(y) \wedge \phi_{prime}(z) \wedge \phi_{div}(y, x)] \rightarrow \phi_{div}(z, x))$.

(6) Δ_1: $\phi_{code}(x, i, c)$ is such that: a codes a sequence with i-th element c iff $\mathfrak{A} \models \phi_{code}[a]$. In this case we simply write $x_i = c$.
We have to define this formula to say that x codes a sequence and if p is the ith prime, i.e. $p = p_i$, then p^{c+1} divides x but p^{c+2} does not:
$\phi_{code}(x, i, c)$ is $\phi_{code}(x) \wedge \exists p \leq x$
$(\phi_{th-prime}(p, i) \wedge (\exists y \leq x)(\exists z < x \cdot x)$
$[\phi_{exp}(p, c+1, y) \wedge \phi_{exp}(p, c+2, z) \wedge \phi_{div}(y, x) \wedge \neg\phi_{div}(z, x)])$.

In the above definition we use the variable y to denote p^{c+1} and the variable z to denote p^{c+2}. Note that since $z = y \cdot p$ and $y \leq x$, we must have that $z < x^2$. Since all of the new quantifiers are bounded the complexity of $\phi_{code}(x, i, c)$ is Δ_1.